A Meshfree Solver for the MEG Forward Problem

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Noninvasive estimation of brain activity via magnetoencephalography (MEG) involves an inverse problem whose solution requires an accurate and fast forward solver. To this aim, we propose the Method of Fundamental Solution (MFS) as a meshfree alternative to the Boundary Element Method (BEM). The solution of the MEG forward problem is obtained, via the Method of Particular Solutions (MPS), by numerically solving a boundary value problem for the electric scalar potential, derived from the quasi-stationary approximation of Maxwell's equations. The magnetic field is then computed by Biot-Savart law. Numerical experiments have been carried out in a realistic head geometry. Comparisons with a state-of-the-art BEM solver shows the attractiveness of the proposed method.

***Index Terms*—MEG, method of fundamental solutions, method of particular solutions, meshfree methods.**

I. Introduction

THE PROBLEM of estimating the sources of neuronal activity in the human brain from electroencephalography (EEG) and magnetoencephalography (MEG) signals is of great interest both in clinical and basic health research. The brain activity generates small electric potential and magnetic field distributions that can be measured by means of an array of electrodes on the scalp, for EEG, or superconducting quantum interference devices (SQUID), for MEG, located near the head.

EEG can detect activity both in the sulci and at the top of the cortical gyri, whereas MEG is most sensitive to activity originating in sulci and provides a better spatial resolution [1].

Starting from a set of measured data (electric potential or magnetic fields), an inverse problem must be solved to estimate the corresponding neuronal activity sources. To this end, an accurate forward solver must be designed as a component in the solution of this inverse problem. Such a numerical tool computes the scalp potential and/or magnetic fields generated by a set of current sources representing the neural activity, given knowledge of both the physical properties of the biological tissues and the geometry of the head [2].

Here we focus the attention on the solution of the MEG forward problem. So far, the M/EEG forward problem has been addressed by traditional mesh-based numerical methods, whose literature is vast [2]. Among these methods, the Boundary Element Method (BEM) [3]–[5] has become the method of choice because of its efficiency with respect to the Finite Elements Method (FEM) [6]–[8], and it is currently implemented in widely used software packages for M/EEG source analysis [9], [10]. However, the BEM involves costly numerical integration, requires an often nontrivial meshing of the domain boundaries at high quality and could potentially introduce mesh-related artifacts in the reconstructed neural activation pattern.

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II. MEG Forward Problem Formulation

Common models rely upon a piecewise-constant conductivity approximation so that the head is described as a volume conductor composed of electrically homogeneous compartments: typically the brain, skull and scalp. One common difficulty with this model is that electric potentials at the scalp are strongly distorted due to the conductivity difference between the tissues composing the head. In contrast, magnetic fields depend on the electrical currents flowing in the high conductivity tissues, i.e., in the brain, with a negligible contribution given by the weak currents flowing in the skull and the scalp. Therefore, while a detailed geometrical model with at least three compartments (brain, skull, scalp) is needed in solving the EEG forward problem, a simple homogeneous model of the high-conductivity brain compartment is sufficient to solve the MEG forward problem [3], [11].

Let Ω be the homogeneous domain that represents the brain, with boundary $\partial\Omega$ and electrical conductivity σ. The volume surrounding Ω can be considered as the ambient air, with negligible electrical conductivity.

It is convenient to express the current density field at a point $\boldsymbol{p} \in \Omega$ as the sum of the source (impressed) current density $\mathbf{J}_s(\boldsymbol{p})$ and the volume current density, i.e.,

$$\mathbf{J}(\boldsymbol{p}) = \mathbf{J}_s(\boldsymbol{p}) - \sigma \nabla \phi(\boldsymbol{p}), \tag{1}$$

where ϕ is the electric scalar potential. We shall concentrate our attention on the simplest case of a single neural source, representable by a current dipole of moment $\mathbf{Q}$ located at $\boldsymbol{p}' \in \Omega$ [1]. What follows can be extended to the case of many dipoles by simple application of the superposition principle.

With this position, the source current density is given by

$$\mathbf{J}_s(\boldsymbol{p}) = \mathbf{Q}\delta(\boldsymbol{p} - \boldsymbol{p}'), \tag{2}$$

where $\delta(\boldsymbol{p} - \boldsymbol{p}')$ is the Dirac delta function centered at the source point $\boldsymbol{p}'$.

The solution of the MEG forward problem involves the solution of a potential problem on the boundary $\partial\Omega$. In fact, in the quasi-stationary approximation of the the Maxwell's equations [1], [12], the following equation holds:

$$\nabla^2 \mathbf{B}(\boldsymbol{p}) = -\mu \nabla \times \mathbf{J}(\boldsymbol{p}), \tag{3}$$

where μ is the magnetic permeability of the medium, supposed to be equal to that of the air, and the current density on the right-hand side is given by Equation (1) if the electric scalar potential at $\boldsymbol{p}$ is known. The solution of Equation (3) under the assumption of a null magnetic field at an infinite distance from the sources, is given by the Biot-Savart law [13]:

$$\mathbf{B}(\boldsymbol{p}) = \frac{\mu}{4\pi} \int_{\Omega} \mathbf{J}(\boldsymbol{p}^*) \times \frac{\boldsymbol{p} - \boldsymbol{p}^*}{\|\boldsymbol{p} - \boldsymbol{p}^*\|^3} \mathrm{d}v(\boldsymbol{p}^*). \tag{4}$$

where $\mathrm{d}v$ is the differential volume element for Ω.

By using Equation (1), the integral above can be split into two parts. One, $\mathbf{B}_s(\boldsymbol{p})$, relates to the contribution of the source current density, and the other describes the contribution of volume current density,

$$\mathbf{B}(\boldsymbol{p}) = \mathbf{B}_s(\boldsymbol{p}) + \frac{\mu}{4\pi}\sigma \int_{\Omega} \nabla\phi(\boldsymbol{p}^*) \times \frac{\boldsymbol{p} - \boldsymbol{p}^*}{\|\boldsymbol{p} - \boldsymbol{p}^*\|^3} \mathrm{d}v(\boldsymbol{p}^*). \tag{5}$$

For a dipole source, the following analytic expression of the first term $\mathbf{B}_s(\boldsymbol{p})$ is known [13]:

$$\mathbf{B}_s(\boldsymbol{p}) = \frac{\mu}{4\pi} \mathbf{Q} \times \frac{\boldsymbol{p} - \boldsymbol{p}'}{\|\boldsymbol{p} - \boldsymbol{p}'\|^3}. \tag{6}$$

The volume integral over Ω in Equation (5) can be transformed into a surface integral over the interfaces by applying a corollary of the Divergence Theorem [1] that yields:

$$\mathbf{B}(\boldsymbol{p}) = \mathbf{B}_s(\boldsymbol{p}) - \frac{\mu}{4\pi}\sigma \int_{\partial\Omega} \phi(\boldsymbol{p}^*)\boldsymbol{n}(\boldsymbol{p}^*) \times \frac{\boldsymbol{p} - \boldsymbol{p}^*}{\|\boldsymbol{p} - \boldsymbol{p}^*\|^3} \mathrm{d}s(\boldsymbol{p}^*), \tag{7}$$

where $\boldsymbol{n}$ is the unit vector normal to the boundary and $\mathrm{d}s$ is the differential surface element for $\partial\Omega$.

The electric scalar potential ϕ in Ω due to a current dipole is governed by a boundary value problem (BVP) for the Poisson equation [13]:

$$\begin{cases} \sigma\nabla^2\phi(\boldsymbol{p}) = \nabla \cdot (\mathbf{Q}\delta(\boldsymbol{p} - \boldsymbol{p}')), & \boldsymbol{p} \in \Omega, \\ \sigma\boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi(\boldsymbol{p}) = 0, & \boldsymbol{p} \in \partial\Omega. \end{cases} \tag{8}$$

Once ϕ is known, the magnetic field at any point outside the head can be evaluated by Equation (7).

III. Methodology

We propose the application of the Method of Fundamental Solutions (MFS) [14] via the Method of Particular Solutions (MPS) for solving the potential problem (8). The MFS approximates the solution u of the given homogeneous BVP by a linear combination of *fundamental solutions* K of the governing homogeneous PDE, i.e.,

$$u(\boldsymbol{p}) \approx \sum_{\xi_j \in \Xi} c_j K(\boldsymbol{p}, \boldsymbol{\xi}_j), \quad \boldsymbol{p} \in \Omega, \tag{9}$$

where Ξ is a set of *centers* located on a *fictitious boundary* outside the physical domain Ω (Figure 1) in order to avoid potential singularities of K in the representation of the solution. The coefficients c_j of the linear combination are determined by enforcing equality of $u(\boldsymbol{p})$ to the boundary conditions at a finite set of collocation points.

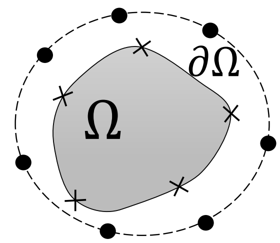

Fig. 1. Collocation points (crosses) and centers (dots) distributed on the physical and fictitious boundaries, respectively.

An inhomogeneous problem can be reduced to a homogeneous one by the MPS, i.e., by considering the solution u as the sum of a particular solution u_p and its associated homogeneous solution u_h.

The governing PDE of the scalar potential problem in Ω is a Poisson equation (see Section II). Let us express the scalar potential function in Ω, by means of the MPS, as

$$\phi(\boldsymbol{p}) = \phi_h(\boldsymbol{p}) + \phi_p(\boldsymbol{p}). \tag{10}$$

An analytical expression for a particular solution ϕ_p of the PDE of the BVP in Ω, when a neural source is located at $\boldsymbol{p}' \in \Omega$, is known [13]:

$$\phi_p(\boldsymbol{p}) = \frac{1}{4\pi\sigma} \frac{\boldsymbol{p} - \boldsymbol{p}'}{\|\boldsymbol{p} - \boldsymbol{p}'\|^3} \cdot \mathbf{Q}. \tag{11}$$

Therefore, the homogeneous term ϕ_h is given by the solution of the following BVP:

$$\begin{cases} \nabla^2\phi_h(\boldsymbol{p}) = 0, & \boldsymbol{p} \in \Omega, \\ \boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi_h(\boldsymbol{p}) = -\boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi_p(\boldsymbol{p}) & \boldsymbol{p} \in \partial\Omega. \end{cases} \tag{12}$$

and it can be approximated, by means of MFS, as a linear combination of fundamental solutions for the 3D Laplace equation:

$$\hat{\phi}_h(\boldsymbol{p}) = \sum_{\xi_j \in \Xi} c_j K(\boldsymbol{p}, \boldsymbol{\xi}_j), \quad \boldsymbol{p} \in \Omega, \tag{13}$$

where Ξ is the set of centers and the fundamental solution for the Laplace equation in three dimensions is $K(\boldsymbol{p}, \boldsymbol{q}) = (4\pi\|\boldsymbol{p} - \boldsymbol{q}\|)^{-1}$.

The only geometric quantities needed to compute the potentials are the normals to the boundary and the distances between the boundary collocation points and the centers; therefore, the proposed method is truly meshfree. Moreover, no costly numerical integration is needed and its implementation is straightforward.

It is worth mention that for certain problems and suitably smooth data and domains, the proposed method has been proved to be exponentially convergent [15]–[17], whereas the convergence rate of BEM and FEM is limited by the maximum degree of the polynomials adopted as basis functions.

IV. Numerical Results

In order to assess the viability of the proposed approach in solving the MEG forward problem for a realistic head geometry, we have carried out a comparison with the BEM state-of-the-art formulation [5], [18].

First we compare the accuracy of the MFS solver and the BEM solver in evaluating the scalar potential on the inner skull surface. A single unitary dipole source is simulated in the brain (with electrical conductivity equal to 0.2 S/m) at roughly 1 cm from the inner skull surface. This choice is appropriate if one considers the location of the real neural sources in the cerebral cortex. Figures 2 and 3 show the potential map for the finest discretization of the solution.

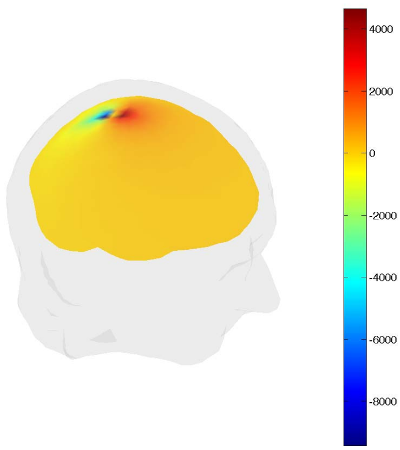

Fig. 2. Electric scalar potential [V] on the inner skull surface for a unitary dipole–results were obtained by the BEM solver with 4500 triangles.

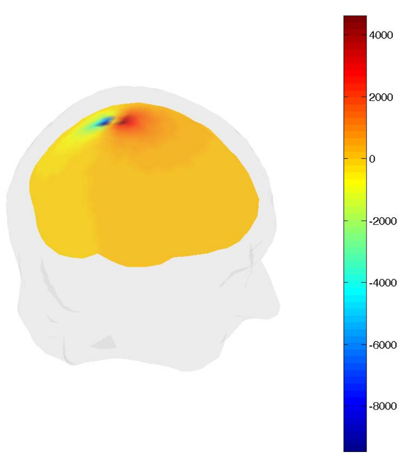

Fig. 3. Electric scalar potential [V] on the inner skull surface for a unitary dipole–results were obtained by the MFS solver with 4500 collocation points and 2250 centers.

Since there is no way of knowing the ground truth solution for realistic geometries, we choose a BEM solution obtained with a fine mesh (4500 triangles) as a reference. The relative 2-norm is adopted to estimate the accuracy with respect to the reference solution, as reported in Table I.

TABLE I
Comparison of the solutions for the potential problem obtained with the proposed MFS solver and the state-of-the-art BEM with respect to a reference solution.

N	MFS Estimated Accuracy	BEM Estimated Accuracy
500	0.32687	0.54554
1500	0.11374	0.17825
2500	0.09295	0.18846
3500	0.08206	0.11594
4500	0.05849	N/A

The results shown in Table I suggest that the MFS solver outperforms the BEM solver. Moreover, the MFS provides a reduction in CPU time that becomes more significant as higher accuracy is requested, since no numerical integration is required in the assembly of the system matrix.

Reusing the same real geometry, we now compare the computed external magnetic fields. In this case, we have assumed a set of $1,000$ unitary dipole sources randomly distributed inside the brain (red points in Figure 4), with random orientations.

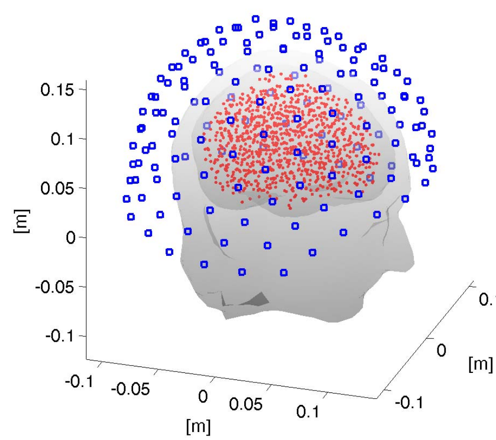

Fig. 4. Real head geometry: red points are dipole sources; blue squares are sensor locations.

The same figure shows the magnetic field evaluation points (blue squares) on a SQUID helmet. Figures 5 and 6 show the magnetic field map obtained with BEM and with MFS, respectively.

The maps depict a good agreement between the two solutions, with a 2-norm relative difference equal to 0.12204.

V. Conclusion

In this paper we have shown that the solution of the MEG forward problem can be sought by the Method of

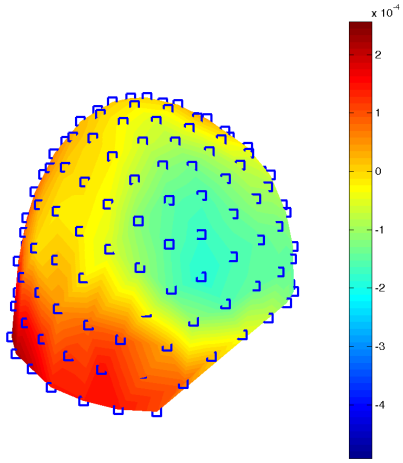

Fig. 5. BEM magnetic field map [T]–1, 000 unitary dipole sources randomly distributed inside the brain with random orientations.

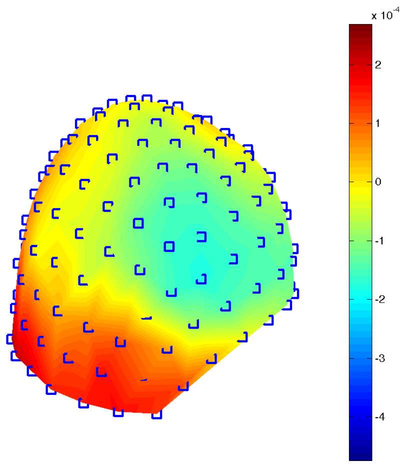

Fig. 6. MFS magnetic field map [T]–1, 000 unitary dipole sources randomly distributed inside the brain with random orientations.

Fundamental Solutions via the Method of Particular Solutions. The proposed method is a boundary-type, integration-free and easy-to-implement alternative to mesh-based methods, such as the widely used Boundary Element Method.

We have successfully compared the proposed approach with the state-of-the-art BEM formulation in solving the MEG forward problem with a realistic head model. The method needs no meshing algorithms in the pre-processing stage – which simplifies the experimental setup – and no numerical quadrature to assemble the system matrix – which improves the computational cost and is important when incorporating the forward solver within the solution of the inverse problem needed for identification of the neural sources.

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